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DETERMINISTIC STRATEGY AND STOCHASTIC PROGRAMS*

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The problem of the control which minimizes the guaranteed result /2/ is solved by the method of stochastic synthesis /1/. The object is described by a linear differential equation. The quality indicator is composed of the phase vector at the end of the process and of integrals of samples of the control and dynamic interference. The information element at the current instant of time t is composed of the signal representing the actual motion with the error and of the control history up to the instant t. The information error, the dynamic interference, and the control are constrained by geometric limitations. This paper is related to papers /3–7/.

1. Statement of the problem. Consider the x-object defined by the differential equation

$$x^{\cdot} = A(t)x + B(t)u + C(t)v, \quad t_0 \leqslant t \leqslant \vartheta \tag{1.1}$$

where x, u, v are column vectors of the matrix function $A(t), B(t), C(t)$ which are continuous. The control u and the disturbances v are constrained by limitations $u \in P, v \in Q$, where P and Q are compacta. Information about the initial and current states $x[t_0] = x_0$ and $x[t], t > t_0$ of the phase vector x are defined by the vectors $x_0^* = x_0 + \Delta x_0^*$ and $q^*[t] = K(t)x[t] - \Delta q^*[t]$, $t_0 < t \leqslant \vartheta$, where $K(t)$ is the continuous metric function, and the distortions Δx_0^* and $\Delta q^*[t]$ are constrained by the limitations $\Delta x_0^* \in R$ and $\Delta q^*[t] \in S(t)$, where R and $S(t)$ are compacta. The sets $S(t)$ are continuous in t in the Hausdorff metric. We denote functions of time as follows $x[t_*[\cdot]t^*] = \{x[\tau], t_* \leqslant \tau \leqslant t^*\}$, $x(t_*[\cdot]t^*] = \{x[\tau], t_* < \tau \leqslant t^*]$, etc. The set

$$Y[t] = \{x_0^*, q^*(t_0[\cdot]t], u(t_0[\cdot]t]\} \tag{1.2}$$

will be used as the information transform of $Y[t]$.

Piecewise-continuous samples $q^*[\tau]$ and measurable samples $u[\tau]$ in (1.2) are admissible Moreover the admissible functions $q^*[\tau]$ and $u[\tau]$ are connected by the following condition. Let $X[\tau, \nu]$ be the fundamental matrix of solutions of the equation $dx/d\tau = A(\tau)x$. We put

$$r[\tau] = q^*[\tau] - K(\tau)\int_{t_0}^{\tau} X[\tau, \nu]B(\nu)u[\nu]\,d\nu \tag{1.3}$$

The vector x_0 and the measurable function $v(t_0[\cdot]t] = \{v[\tau] \in Q, t_0 < \tau \leqslant t\}$ that satisfy the imbedding

$$(x_0^* - x_0) \in R \tag{1.4}$$

$$(r[\tau] - K(\tau)[X[\tau, t_0]x_0 - \int_{t_0}^{\tau} X[\tau, \nu]C(\nu)v[\nu]\,d\nu]) \in S(\tau), \quad t_0 < \tau \leqslant t \tag{1.5}$$

must exist.

We shall call the strategy $u(\cdot)$ the function $u(t, Y, \varepsilon)$ defined for $t \in [t_0, \vartheta)$, $\varepsilon > 0$ and all possible transforms $Y = Y[t]$. Here ε is the exactness parameter /1, 2, 7/. Suppose the instant $t_* \in [t_0, \vartheta)$ has occurred and the transform $Y[t_*]$ has been realized. The law of control U on the remaining segment $[t_*, \vartheta]$ is the set of three fixed components $U = \{u(\cdot), \varepsilon, \Delta\{t_i\}\}$, where $\Delta\{t_i\}$ is the partitioning of segment $[t_*, \vartheta]: t_1 = t_*; t_{i+1} > t_i, i = 1, \ldots, k$ and $t_{k+1} = \vartheta$, k is a positive integer. The law U and the transform $Y[t_*]$ generate the continuation of the motion $x[t_*[\cdot]\vartheta]$. It is the solution of the stepwise equation

$$x^{\cdot}[t] = A(t)x[t] + B(t)u(t_i, Y[t_i], \varepsilon) + C(t)v[t], \quad t_i < t \leqslant t_{i+1}, \quad i = 1, \ldots, k \tag{1.6}$$

which admits of measurable samples $v[t] \in Q$. The process quality indicator is specified by

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$$\gamma = |x[\vartheta]| + \int_{t_0}^{\vartheta} [\varphi(t, u[t]) + \psi(t, v[t])]\, dt \tag{1.7}$$

where the functions φ and ψ are continuous and the symbol $|x|$ denotes the Euclidean norm of the vector x.

We define the guaranteed result for the law U by the equation

$$\rho(t_*, Y[t_*]; U) = \sup_{q^*(t_*[\cdot]\vartheta]} \; \sup_{x_0, v(t_0[\cdot]\vartheta])} \gamma \tag{1.8}$$

where the internal upper face is calculated, using all possible pairs $\{x_0, v(t_0[\cdot]\vartheta]\}$ that satisfy conditions (1.4) and (1.5), when $t = \vartheta$. The external upper face is calculated using all possible continuations $q^*(t_*[\cdot]\vartheta]$ of the admissible component $q^*(t_0[\cdot]t_*]$ from $Y[t_*]$. The admissibility of the continuation $q^*(t_*[\cdot]\vartheta]$ is determined recurrently by steps $t_i < t \leqslant t_{i+1}$ paired with the continuation $u(t_*[\cdot]\vartheta]$ of the sample of the control $u[\tau] = u(t_i, Y[t_i], \varepsilon)$, $t_i < \tau \leqslant t_{i+1}$, $i = 1, \ldots, k$. Here $Y[t_i]$ is the transform composed of the component x_0^* specified at the beginning and of admissible components $q^*(t_0[\cdot]t_i]$ and $u(t_0[\cdot]t_i]$ referred to the instant t_i.

We define the guaranteed result for the strategy $u(\cdot)$ by the equation

$$\rho(u(\cdot); t_*, Y[t_*]) = \overline{\lim_{\varepsilon\to 0}} \lim_{\delta\to 0} \sup_{U_\delta} \rho(t_*, Y[t_*]; U_\delta) \tag{1.9}$$

where U_δ is the law whose step $\max_i (t_{i+1} - t_i)$ does not exceed $\delta > 0$.

We have to determine the optimal strategy $u^\circ(\cdot)$ that satisfies the condition

$$\rho(u^\circ(\cdot); t_*, Y[t_*]) = \min_{u(\cdot)} \rho(u(\cdot); t_*, Y[t_*]) = \rho^\circ(t_*, Y[t_*]) \tag{1.10}$$

whatever the instant $t_* \in [t_0, \vartheta]$ and the admissible transform $Y[t_*]$. The optimal strategy $u^\circ(\cdot)$ does exist. We call the quantity $\rho^\circ(t_*, Y[t_*])$ the optimal guaranteed result. Below, we describe the calculation of $\rho^\circ(t_*, Y[t_*])$ and the derivation of $u^\circ(\cdot)$ using the method of programmed stochastic synthesis /1, 2, 7/.

2. The programmed stochastic derivation. Let us construct the auxiliary model Z whose basis is a z object which operates in imaginary time τ. The current state of the z object is defined by its phase vector $z = \{w, \zeta\}$, where the dimensions of the vector w is the same as of the vector x, and ζ is a scalar. The variation of w and ζ are subject to the equations

$$w^{\cdot} = A(\tau) w + B(\tau) u^* + C(\tau) v^* \tag{2.1}$$

$$\zeta^{\cdot} = \varphi(\tau, u^*) + \psi(\tau, v^*) \tag{2.2}$$

where u^* and v^* are constrained by the restrictions $u^* \in P$ and $v^* \in Q$.

Let the instant $\tau_* \in [t_0, \vartheta]$ be specified. We set the partitioning $\Delta\{\tau_j\}: \tau_1 = \tau_*, \tau_{j+1} > \tau_j$, $j = 1, \ldots, g$; $\tau_g = \vartheta$ with g a natural number, and introduce the probability space /8/ $\{\Omega, B, p\}$, $\omega = \{\xi_1, \ldots, \xi_g\}$ generated by the random quantities ξ_j independent in the aggregate, each of which $\xi_j = \xi[\tau_j]$ is realized at its own instant τ_j. All ξ_j are equally probable in the segment /0, 1/.

We will call $u^*(\cdot)$ the stochastic program for the control of the non-anticipatory function /8/

$$u^*(\tau, \omega) = u^*[\tau, \xi_1, \ldots, \xi_j] \in P, \quad \tau_j < \tau \leqslant \tau_{j+1}, \quad j = 1, \ldots, g-1$$

and the stochastic program $v^*(\cdot)$ for the dynamic disturbance the random function

$$v^*(\tau, \omega) = v^*[\tau, \xi_1, \ldots, \xi_g] \in Q$$

that is measurable over the set of variables τ, ω.

Let some deterministic sample $u^*(t_0[\cdot]\tau_*] = \{u^*[\tau] \in P,\ t_0 < \tau \leqslant \tau_*\}$ that is measurable be indicated. The random vector $w(\cdot) = \{w(\omega), \omega \in \Omega_0\}$, the number ζ_0, the sample $u^*(t_0[\cdot]\tau_*]$, and the pair of programs $u^*(\cdot)$, $v^*(\cdot)$ define the random motion $z(\cdot,\cdot) = \{z(\tau, \omega) = \{w(\tau, \omega), \zeta(\tau, \omega)\}, t_0 \leqslant \tau \leqslant \vartheta, \omega \in \Omega: w(t_0, \omega) = w(\omega), \zeta(t_0, \omega) = \zeta_0\}$ of the z object. The functions $w(\tau, \omega)$ and $\zeta(\tau, \omega)$ are here solutions of the stochastic differential equations (2.1) and (2.2), when $u^* = u^*[\tau]$, $t_0 < \tau \leqslant \tau_*$, $u^* = u^*(\tau, \omega)$, $\tau_* < \tau \leqslant \vartheta$; $v^* = v^*(\tau, \omega)$, $t_0 < \tau \leqslant \vartheta$.

As in (1.3), we assume the dimensions of the vector r to be the same as of the vector q^* from (1.2). Let some set of admissible components x_0^*, $q^*(t_0[\cdot]\tau_*]$, $u(t_0[\cdot]\tau_*]$, that constitute some transform $Y[\tau_*]$ (1.2), when $t = \tau_*$), be indicated. Let $r(t_0[\cdot]\tau_*]$ be the respective function (1.3). We call $r^*(\cdot)$ the stochastic program admissible for the given x_0^*, $r(t_0[\cdot]\tau_*]$ and $u(t_0[\cdot]\tau_*]$ for the non-anticipatory function

$$r^*(\tau, \omega) = r^*[\tau, \xi_1, \ldots, \xi_j], \quad \tau_j < \tau \leqslant \tau_{j+1}, \quad j = 1, \ldots, g-1$$

whose samples are almost certainly piecewise continuous and satisfy (also almost continuously) the following conditions:

$$(r^*(\tau, \omega) - K(\tau)[X[\tau, t_0]w(\omega) + \int_{t_0}^{\tau} X[\tau, \nu]C(\nu)v^*(\nu, \omega)d\nu]) \in S(\tau), \quad \tau_* < \tau \leqslant \vartheta \tag{2.3}$$

and the pair $\{w(\omega), v^*(\nu, \omega)\}$ consisting of the random vector $w(\omega)$ and the stochastic program $v^*(\nu, \omega)$ satisfy, in addition to (2.3), the conditions

$$(x_0^* - w(\omega)) \in R \tag{2.4}$$

$$(r(\tau) - K(\tau)[X[\tau, t_0]w(\omega) + \int_{t_0}^{\tau} X[\tau, \nu]C(\nu)v^*(\nu, \omega)d\nu]) \in S(\tau), \quad t_0 < \tau \leqslant \tau_* \tag{2.5}$$

To each admissible set x_0^*, $r(t_0[\cdot]\tau_*]$, $u(t_0[\cdot]\tau_*]$ there corresponds at least one admissible program $r^*(\cdot) = \{r^*(\tau, \omega), \tau_* < \tau \leqslant \vartheta, \omega \in \Omega\}$.

We define the state of model Z at the instant τ_* by its phase element

$$Z[\tau_*] = \{x_0^*, r(t_0[\cdot]\tau_*], z_*\} \tag{2.6}$$

where $r(t_0[\cdot]\tau_*]$ is an admissible function, i.e. for which the pair $\{w(\omega), v^*(\nu, \omega)\}$ exists that satisfies conditions (2.4) and (2.5), and the component z_* is some vector $z_* = \{w_*, \zeta_*\}$.

Let us estimate the element $Z[\tau_*]$ by a suitable indicator $\rho(\tau_*, Z[\tau_*]; \Delta\{\tau_j\})$ as follows. Let l be a vector of the same dimensions as the vector w. We introduce a random vector $l(\cdot) = \{l(\omega), \omega \in \Omega\}$ with the norm

$$\|l(\cdot)\| = \operatorname{vrai}\max_{\omega \in \Omega} |l(\omega)| \tag{2.7}$$

Let $M\{\ldots\}$ denote the expectation and $M\{\ldots|\xi_1, \ldots, \xi_j\}$ the conditional expectation, where the prime denotes transposition. We introduce the quantity

$$\begin{aligned} &\varkappa(\tau_*, Z[\tau_*], \Delta\{\tau_j\}, l(\cdot)) = \\ &\sup_{r^*(\cdot)} \sup_{\{w(\cdot), v^*(\cdot)\}} \inf_{u^*(\cdot)} M\{l'(\omega)[X[\vartheta; t_0]w(\omega) - w_*] + \\ &\zeta_* + \int_{\tau_*}^{\vartheta} [l'(\omega)X[\vartheta, \tau]B(\tau)u^*(\tau, \omega) - \varphi(\tau, u^*(\tau, \omega))]d\tau + \\ &\int_{t_0}^{\vartheta} [l'(\omega)X[\vartheta, \tau]C(\tau)v^*(\tau, \omega) - \psi(\tau, v^*(\tau, \omega))]d\tau\} \end{aligned} \tag{2.8}$$

Here the inner upper face is calculated using all pairs $\{w(\omega), v^*(\tau, \omega)\}$ that satisfy conditions (2.3)–(2.5) for a fixed admissible program $r^*(\cdot)$. The external upper face is then calculated using all programs $r^*(\cdot)$ that are admissible for component x_0^* and $r(t_0[\cdot]\tau_*]$ from $Z[\tau_*]$. The lower face in (2.8) is calculated using all possible stochastic programs $u^*(\cdot)$.

Let us determine the programmed extremum ρ, which evaluates the element $Z[\tau_*]$ for the specified partitioning $\Delta\{\tau_j\}$ by the equation

$$\rho(\tau_*, Z[\tau_*], \Delta\{\tau_j\}]) = \sup_{\|l(\cdot)\| \leqslant 1} \varkappa(\tau_*, Z[\tau_*], \Delta\{\tau_j\}, l(\cdot)) \tag{2.9}$$

If in (2.8) we set

$$w_* = \int_{t_0}^{\tau_*} X[\vartheta, \tau]B(\tau)u[\tau]d\tau, \quad \zeta_* = \zeta_0 + \int_{t_0}^{\tau_*} \varphi(\tau, u[\tau])d\tau \tag{2.10}$$

the quantity $\rho(\tau_*, Z[\tau_*], \Delta\{\tau_j\})$ takes the form

$$\rho(\tau_*, Z[\tau_*], \Delta\{\tau_j\}) = \sup_{\|l(\cdot)\| \leqslant 1} \sup_{r^*(\cdot)} \sup_{\{w(\cdot), v^*(\cdot)\}} \inf_{u^*(\cdot)} M\{l'(\omega)w(\vartheta, \omega) + \zeta(\vartheta, \omega)\}$$

where $z(\tau, \omega) = \{w(\tau, \omega), \zeta(\tau, \omega)\}$ is the corresponding random motion of the z-object generated from the initial state $z(t_0, \omega) = \{w(\omega), \zeta_0\}$ by the programs $v^*(\cdot) = \{v^*(\tau, \omega), t_0 < \tau \leqslant \vartheta\}$ and $u^*(\cdot) = \{u[\tau], t_0 < \tau \leqslant \tau_*; u(\tau, \omega), \tau_* < \tau \leqslant \vartheta\}$.

3. Stability of the program extremum. We put $l_* = M\{l(\cdot)\}$. and take some maximizing sequence for (2.9) of random vectors $l^{(s)}(\cdot)$, $s = 1, 2, \ldots$ for which the following limit exists:

$$\lim_{s\to\infty} l_*^{(s)} = \lim_{s\to\infty} M\{l^{(s)}(\cdot)\} = l^c[\tau_*] \tag{3.1}$$

Let us compose the set $L^c[\tau_*, Z[\tau_*], \Delta\{\tau_j\}]$ of all vectors $l^c[\tau_*]$ that can be obtained by passing to the limit (3.1). The set $L^c[\tau_*, Z[\tau_*], \Delta\{\tau_j\}]$ proves to be closed and convex. These sets are upper continuous and vary on the inclusion, when the component z_* in $Z[\tau_*]$ (2.6) is changed. We select the instant $\tau^* = \tau_i$, where i is some fixed subscript from the numbers $j = 2, \ldots, g$ for fixed τ_*. $\Delta\{\tau_j\}$ and $Z[\tau_*]$. Let the vector h have the dimensions of the vector z. We introduce the sets

$$H(\tau) = \mathrm{co}\,[h : h = \{B^*(\tau)\, u, \varphi(\tau, u)\},\ u \in P],\quad \tau_* < \tau \leqslant \tau^*$$

where the symbol $[\ldots]$ denotes a convex envelope of the set $[\ldots] B^*(\tau) = X[\vartheta,\tau]B(\tau)$. Let us take the element

$$Z[\tau^*] = \{x_0^*, r(t_0[\cdot]\tau^*], z^*\} \tag{3.2}$$

where the admissible function $r(t_0[\cdot]\tau^*]$ is a continuation of the function $r(t_0[\cdot]\tau_*]$ from $Z[\tau_*]$, and vector z^* is connected with the vector z_* from $Z[\tau_*]$ by the equation

$$z^* = z_* - \int_{\tau_*}^{\tau^*} h[\tau]\, d\tau \tag{3.3}$$

and $h[\tau]$ is some fixed measurable function that satisfies the condition

$$h[\tau] \in H(\tau),\quad \tau_* < \tau \leqslant \tau^* \tag{3.4}$$

Let the partitioning $\Delta\{\tau_j^*\}$ for the segment $[\tau^*, \vartheta]$ be linked with the original partitioning $\Delta\{\tau_j\}$ for the segment $[\tau_*, \vartheta]$ by the relation $\tau_j^* = \tau_{j-i-1}$, $j = 1, \ldots, g - i + 1 = g^*$. following the reasoning in /2/, we obtain the estimate

$$\rho(\tau^*, Z[\tau^*], \Delta\{\tau_j^*\}) \leqslant \rho(\tau_*, Z[\tau_*], \Delta\{\tau_j\}) + \int_{\tau_*}^{\tau^*} (s^c[\tau^*]\, h[\tau] - \min_{h\in H(\tau)} s^c[\tau^*]\, h)\, d\tau,\quad s^c = \{l^c, 1\} \tag{3.5}$$

where $l^c[\tau^*]$ is any vector from the set $L^c[\tau^*, Z[\tau^*], \Delta\{\tau_j^*\}]$. We introduce the functional sets

$$H_l = [h(\tau_*[\cdot]\tau^*] : h[\tau] \in H(\tau),\ sh[\tau] = \min_{h\in H(\tau)} sh],\ s = \{l, 1\}$$

where $h(\tau_*[\cdot]\tau^*]$ will be treated as elements of the space of the functions $\{h[\tau], \tau_* < \tau \leqslant \tau^*\}$ with the integrable square with the norm

$$\| h(\tau_*[\cdot]\tau^*] \|_{(2)} = \Big(\int_{\tau_*}^{\tau^*} |h[\tau]|^2\, d\tau\Big)^{1/2}$$

The sets H_l are convex, weakly closed, and weakly upper semicontinuous on the inclusion, when the vector l varies. Using the properties of the sets $L^c[\tau^*, Z[\tau^*], \Delta\{\tau_j^*\}]$ and H_l we can, using the theorem about the fixed point in /9/, establish the existence of the pair $\{l^c[\tau^*], h^c(\tau_*[\cdot]\tau^*]\}$ from which the following conditions are satisfied:

$$l^c[\tau^*] \in L^c[\tau^*, Z^c[\tau^*], \Delta\{\tau_j^*\}] \tag{3.6}$$

$$h^c(\tau_*[\cdot]\tau^*] \in H_{l[\tau^*]} \tag{3.7}$$

where the symbol $Z^c[\tau^*]$ denotes the element $Z[\tau^*]$ (3.2) whose third component is the vector

$$z^{*c} = z_* - \int_{\tau_*}^{\tau^*} h^c[\tau]\, d\tau$$

The theorem on the fixed point mentioned above is applied here to the appropriate representation of pairs $\{l, h(\tau_*[\cdot]\tau^*]\}$. and is constructed in the manner used in similar cases (see, e.g., /2, 10/). Selecting in (3.5) the vector $l^c[\tau^*]$ and the functions $h[\tau] = h^c[\tau]$. we shall prove from (3.6) and (3.7) the validity of the statement given below.

Let the partitioning $\Delta\{\tau_j\}$ of the segment $[\tau_*, \vartheta] \subset [t_0, \vartheta]$ be specified and let the admissible element $Z[\tau_*]$ be selected. We set any arbitrary instant $\tau^* = \tau_i \in \Delta\{\tau_j\}$. Then for admissible continuation $r(t_0[\cdot]\tau^*]$ **of the components** $r(t_0[\cdot]\tau_*]$ from $Z[\tau_*]$ the function $h^c(\tau_*[\cdot]\tau^*]$ can be found from (3.4) that satisfies the following condition. When it is substituted into (3.3), we obtain the vector z^{*c} which, together with the vector x_0^* from

$Z[\tau_*]$, and the admissible continuation function $r(t_0[\cdot]\tau^*]$ indicated above constitute the element $Z^c[\tau^*]$ which satisfies the inequality

$$\rho(\tau^*, Z^c[\tau^*], \Delta\{\tau_j^*\}) \leqslant \rho(\tau_*, Z[\tau_*], \Delta\{\tau_j\}) \tag{3.8}$$

This statement is the expression of the property of the program extremum ρ (2.9), which we shall call the stability of the quantity ρ.

4. The extremal displacement. Consider some transform $Y[t]$ (1.2). Let y be a vector whose dimensions are one greater than the dimensions of the vector x. Hence the vectors y and z are of the same dimensions. The vector function $y[t_0[\cdot]t]$ which we define by the equation

$$y[\tau] = \int_{t_0}^{\tau} \left\| \begin{matrix} X[\vartheta, \nu] B(\nu) u[\nu] \\ \varphi(\nu, u[\nu]) \end{matrix} \right\| d\nu, \quad t_0 \leqslant \tau \leqslant t \tag{4.1}$$

corresponds to the transform $Y[t]$. We shall call the controllable component of the transform $Y[t]$, and the function $r(t_0[\cdot]t] = \{r[\tau], t_0 < \tau \leqslant t\}$ defined in (1.3) the information component of that transform. For the given transform $Y[t]$ we shall call the element $Z[t]$ (2.6) a corresponding element (where $\tau_* = t$)) in which the vector x_0^* and the component $r(t_0[\cdot]t]$ are the same as the component x_0^* and the information component $r(t_0[\cdot]t]$ (1.3) of the transform $Y[t]$. We set the instant $\tau^* \in (t, \vartheta]$ that satisfies the inequality $\tau^* - t \leqslant \delta$, where δ is some positive number. The vector $u_e \in P$ is selected on the basis of the condition

$$(y[t] - z_*)' \left\| \begin{matrix} B^*(t) u_e \\ \varphi(t, u_e) \end{matrix} \right\| = \min_{u \in P} (y[t] - z_*)' \left\| \begin{matrix} B^*(t) u \\ \varphi(t, u) \end{matrix} \right\| \tag{4.2}$$

where $y[t]$ is the value of the controllable component (4.1) of the given image $Y[t]$, and z_* is the component of some element $Z[t]$, which corresponds to that image $Y[t]$. We shall call (4.2) the condition of extremal shift on $Z[t]$.

Let $Y[t]$ and the element $Z[t]$ corresponding to it, be fixed. We continue the third component $u(t_0[\cdot]t]$ of the transform $Y[t]$ (1.2) up to the function $u(t_0[\cdot]\tau^*]$, setting $u[\tau] = u_e, t < \tau \leqslant \tau^*$. Let us continue somehow the information signal $q^*(t_0[\cdot]t]$ from $Y[t]$ up to the signal $q^*(t_0[\cdot]\tau^*]$ which is admissible together with x_0^* from $Y[t]$ and the new component $u(t_0[\cdot]\tau^*]$. In other words, we continue the signal $q^*(t_0[\cdot]t]$ so as to obtain the admissible function

$$r[\tau] = q^*[\tau] - K(\tau) \int_{t_0}^{\tau} X[\tau, \nu] B(\nu) u[\nu] d\nu, \quad t_0 < \tau \leqslant \tau^* \tag{4.3}$$

The vector x_0^* from $Y[t]$ together with the indicated continuations $u(t_0[\cdot]\tau^*]$ and $q^*(t_0[\cdot]\tau^*]$ constitute the new admissible transform $Y[\tau^*]$. Its controllable component $y[t_0[\cdot]\tau^*]$ continues the component $y[t_0[\cdot]\tau_*]$ from $Y[t]$ in accordance with the equation

$$y[\tau] = y[t] - \int_{t}^{\tau} \left\| \begin{matrix} X[\vartheta, \nu] B(\nu) u[\nu] \\ \varphi(\nu, u[\nu]) \end{matrix} \right\| d\nu, \quad t < \tau \leqslant \tau^*$$

Let us take some element $Z[\tau^*] = \{x_0^*, r(t_0[\cdot]\tau^*], z^*\}$ which corresponds to the new transform $Y[\tau^*]$ and such that the component $z^* = z[\tau^*]$ is defined by (3.3), (where $\tau_* = t$) and $h[\tau]$ is some function that satisfies condition (3.4).

The following statement holds. Let

$$\lambda = \max_{t_0 \leqslant t \leqslant \vartheta} \|A(t)\| - 1 = \max_{t_0 \leqslant t \leqslant \vartheta} \max_{|x|=1} |A(t)x| - 1 \tag{4.4}$$

For any number $\alpha > 0$ it is possible to indicate a number $\delta > 0$ such that whatever the admissible function $r(t_0[\cdot]\tau^*]$ and the function $h(t[\cdot]\tau^*]$ in (4.3), the following inequality is satisfied

$$|y[\tau^*] - z^*|^2 \leqslant |y[t] - z_*|^2 \exp 2\lambda(\tau^* - t) + \alpha(\tau^* - t) \tag{4.5}$$

provided $\tau^* - t \leqslant \delta$.

5. The limit program extremum. The validity of the following statement is established on the basis of the assertions in Sects.3 and 4.

Whatever the elements $Z[\tau_*]$ and the sequence of divisions $\Delta^{(s)} = \Delta\{\tau_j^{(s)}\}$, $s = 2, 3, \ldots$ of the segment $[\tau_*, \vartheta]$ that satisfies the condition

$$\lim_{s \to \infty} \max_j (\tau_{j+1}^{(s)} - \tau_j^{(s)}) = 0 \tag{5.1}$$

there exists the limit

$$\lim_{s\to\infty} \rho(\tau_*, Z[\tau_*], \Delta\{\tau_j^{(s)}\}) = \rho(\tau_*, Z[\tau_*]) \tag{5.2}$$

The limit $\rho(\tau_*, Z[\tau_*])$ in (5.2) is one and the same for any sequence $\Delta\{\tau_j^{(s)}\}$ that satisfies condition (5.1). The quantity $\rho(\tau_*, Z[\tau_*])$ has the same stability property as the quantity $\rho(\tau_*, Z[\tau_*], \Delta\{\tau_j\})$. We can select any instant $\tau^* \in (\tau_*, \vartheta]$ for the instant τ^* appearing in the stability condition. We call $\rho(\tau_*, Z[\tau_*])$ the limit program extremum.

From the stability property $\rho(\tau_*, Z[\tau_*])$ and from the property (4.5) of extremal displacement (4.2) given in Sect.4 there follows the proof of the construction of the optimal strategy $u^\circ(t, Y, \varepsilon)$. Let us take some information transform $Y[t]$ to which there corresponds the set of respective elements $Z[t]$ which differ between themselves only by the component z_*. Among the respective elements $Z[t]$ we select the element $Z^{(c)}[t]$ which accompanies the transform $Y[t]$ and satisfies the condition

$$\rho(t, Z^{(c)}[t]) = \min_{Z[t]} \rho(t, Z[t])$$

with the constraint

$$|y[t] - z_*| \leqslant \varepsilon \exp 2\lambda(t - t_0)$$

where λ is a number from (4.4). Let $z^{(c)}[t]$ be a component of the accompanying element $Z^{(c)}[t]$. We specify the vector $u_\varepsilon(t, Y[t], \varepsilon)$ that determines the extremal strategy $u_\varepsilon(t, Y, \varepsilon)$, on the basis of the condition of extremal displacement (4.2), where $z_* = z^{(c)}[t]$.

We conclude from the estimate (4.5) and the stability of the extremum $\rho(t, Z[t])$ that for any initial transform $Y[t_*]$ the control law U, based on the strategy $u_\varepsilon(\cdot)$ and working with a partitioning $\Delta\{t_i\}$ with a fairly small step $\max_i (t_{i-1} - t_i) \leqslant \delta$, guarantees for the transform obtained the inequalities

$$|u[\vartheta] - z^{(c)}[\vartheta]| \leqslant \mu(\varepsilon, \delta), \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \mu(\varepsilon, \delta) = 0 \tag{5.3}$$

$$\rho(\vartheta, Z^{(c)}[\vartheta]) \leqslant \rho(t_*, Z^{(c)}[t_*]) \tag{5.4}$$

By definition of the quantity $\rho(\vartheta, Z^{(c)}[\vartheta])$ we have the equation

$$\rho(\vartheta, Z^{(c)}[\vartheta]) = \max_{\|l(\cdot)\|\leqslant 1} \max_{\{w(\cdot), v(\cdot)\}} M\Big\{l'(\omega)(X[\vartheta, t_0]\, w(\omega) + w^{(c)}[\vartheta]) - \zeta^{(c)}[\vartheta] + \int_{t_0}^{\vartheta} [l'(\omega) X[\vartheta, \tau] C(\tau) v(\tau, \omega) + \psi(\tau, v(\tau, \omega))]\, d\tau\Big\} \tag{5.5}$$

where $w^{(c)}[\vartheta]$ and $\zeta^{(c)}[\vartheta]$ are components of vector $z^{(c)}[\vartheta]$ of the accompanying element $Z^{(c)}[\vartheta]$. From (5.3) and (5.5) we have the estimate

$$\rho(\vartheta, Z^{(c)}[\vartheta]) \geqslant \max_{\{x_0, v[\cdot]\}} (|x[\vartheta]| - \int_{t_0}^{\vartheta} [\varphi(\tau, u[\tau]) + \psi(\tau, v[\tau])]\, d\tau) + \sigma(\varepsilon, \delta) \tag{5.6}$$

where

$$x[\vartheta] = X[\vartheta, t_0]\, x_0 + \int_{t_0}^{\vartheta} X[\vartheta, \tau](B(\tau) u[\tau] + C(\tau) v[\tau])\, d\tau$$

and $u[\tau]$ are values of component $u(t_0[\cdot]\vartheta)$ from $Y[\vartheta]$. The maximum in (5.6) is taken over all pairs $\{x_0, v[\cdot]\}$ that satisfy conditions (1.4) and (1.5) when $t = \vartheta$. The quantity $\sigma(\varepsilon, \delta)$ satisfies the condition

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \sigma(\varepsilon, \delta) = 0$$

From (5.4) and (5.6) by the definition of the factor γ (1.7), and the guaranteed results ρ (1.8), (1.9) we conclude that for the extremal strategy $u_\varepsilon(t, Y, \varepsilon)$ the following inequality holds:

$$\rho(u_\varepsilon(\cdot); t_*, Y[t_*]) \leqslant \rho(t_*, Z[t_*]) \tag{5.7}$$

where $Z[t_*]$ is the element that corresponds to the original transform $Y[t_*]$, and the vector $z_* = \{w_*, \zeta_*\}$ from $Z[t_*]$ is defined by (2.10), when $\tau_* = t_*$, $\zeta_0 = 0$.

A further check shows that for any admissible strategy $u(\cdot)$ the inequality

$$\rho(u(\cdot); t_*, Y[t_*]) \geqslant \rho(t_*, Z[t_*]) \tag{5.8}$$

where $Z[t_*]$ is the same corresponding element, must be satisfied.

It follows from (5.7) and (5.8) that $u_\varepsilon(t, Y, \varepsilon)$ is the optimal strategy $u^\circ(t, Y, \varepsilon)$, and the following equation holds:

$$\rho^\circ(t_*, Y[t_*]) = \rho(t_*, Z[t_*]) \tag{5.9}$$

where $Z[t_*]$ is the element that corresponds to the transform $Y[t_*]$ and satisfies the condition that its component z_* is equal to the control component $y[t_*]$ from $Y[t_*]$.

6. The optimal control algorithm. Let the initial information transform $Y[t_0] = x_0^*$ be specified, and an arbitrarily small number $\alpha > 0$ be selected beforehand. It follows from Sects.3–5 that to realize the law of control U which would guarantee the inequality

$$\gamma \leqslant \rho^\circ(t_0, Y[t_0]) + \alpha \tag{6.1}$$

one has to proceed as follows. First, a reasonably small number $\varepsilon > 0$ is to be selected and the partitioning $\Delta\{t_i\}$, $i = 1, \ldots, k$ of the segment $[t_0, \vartheta]$ with a fairly small step δ is to be chosen. Next, for the recurrent instant t_i at which the transform $Y[t_i]$ is realized, it is necessary to specify the partitioning $\Delta\{\tau_j\}$, $j = 1, \ldots, k - i + 1$ of the next segment $[t_i, \vartheta]$ which satisfies the condition $\tau_j = t_{i+j-1}$, $j = 1, \ldots, g$. Finally, the function $\rho(t_i, Z[t_i], \Delta\{\tau_j\})$ is to be calculated for elements $Z[t_i]$ that correspond to the image $Y[t_i]$ and satisfy the condition

$$|z_*[t_i] - y[t_i]| \leqslant \varepsilon \exp 2\lambda(t_i - t_0) \tag{6.2}$$

From (2.8) and (2.9) we have

$$\rho(t_i, Z[t_i], \Delta\{\tau_j\}) = \tag{6.3}$$
$$\sup_{\|l(\cdot)\|\leqslant 1}\ \sup_{r^*(\cdot)}\ \sup_{\{u(\cdot),\, v^*(\cdot)\}}\ \inf_{u^*(\cdot)} M\Big\{l'(\omega)X[\vartheta, t_0]u(\omega) +$$
$$\int_{t_0}^{\vartheta}[l'(\omega)X[\vartheta,\tau]C(\tau)v^*(\tau,\omega) + \psi(\tau, v^*(\tau,\omega))]\,d\tau +$$
$$\int_{t_i}^{\vartheta}[l'(\omega)X[\vartheta,\tau]B(\tau)u^*(\tau,\omega) + \varphi(\tau, u^*(\tau,\omega))]\,d\tau +$$
$$l'(\omega)u_*[t_i] - \zeta[t_i]\Big\}$$

We represent the random vector $l(\cdot)$ in the form $l(\omega) = M\{l(\omega)\} + a(\omega) = l_* + a(\omega)$, which enables us to determine the upper face ρ (6.3) in the following manner. We fix some vector l_* with the Euclidean norm $|l_*| \leqslant 1$, and calculate the upper face on the right side of (6.3) with the supplementary condition $M\{l(\omega)\} = l_*$. We denote it by $\rho(t_i, Z[t_i], \Delta\{\tau_j\}, l_*)$. We now have

$$\rho(t_i, Z[t_i], \Delta\{\tau_j\}) = \max_{|l_*|\leqslant 1} \rho(t_i, Z[t_i], \Delta\{\tau_j\}, l_*)$$

Then, it is necessary to determine the accompanying element $Z^{(c)}(t_i)$ with component $z^{(c)}[t_i]$, starting from the condition

$$\rho(t_i, Z^{(c)}[t_i], \Delta\{\tau_j\}) = \min_{Z[t_i]} \rho(t_i, Z[t_i], \Delta\{\tau_j\}) = \tag{6.4}$$
$$\min_{z_*[t_i]}\ \max_{|l_*|\leqslant 1} \rho(t_i, Z[t_i], \Delta\{\tau_j\}, l_*)$$

The important fact is that the function ρ on the right side of (6.4) is linear in $z_*[t_i]$ and concave in l_*. Hence the operations min and max in (6.4) can be transposed. We thus obtain the equation

$$\rho(t_i, Z^{(c)}[t_i], \Delta\{\tau_j\}) = \max_{|l_*|\leqslant 1}\ \min_{z_*[t_i]} \rho(t_i, Z[t_i], \Delta\{\tau_j\}, l_*) \tag{6.5}$$

Calculation shows that the component $z^{(c)}[t_i] = \{u^{(c)}[t_i], \zeta^{(c)}[t_i]\}$ of the accompanying element $Z^{(c)}[t_i]$, defined by the condition for a minimum in (6.5) with constraint (6.2) has the form

$$u^{(c)}[t_i] = x[t_i] + l_*\Lambda, \quad \zeta^{(c)}[t_i] = \eta[t_i] + \Lambda \tag{6.6}$$
$$\Lambda = \frac{\varepsilon \exp 2\lambda(t_i - t_0)}{(1 + |l_*|^2)^{1/2}}$$

where $x[t_i]$ and $\eta[t_i]$ are components of the control component $y[t_i]$ of the transform $Y[t_i]$. Moreover it appears that the function of l_* under the maximum sign in (6.5) is strictly concave in l_*. This means that the problem of the maximum in (6.5) has a unique solution $l^\circ[t_i]$. The accompanying element $Z^{(c)}[t_i]$ for the selected ε and for the transform $Y[t_i]$ is uniquely defined. The required vector $u_\varepsilon(t_i, Y[t_i], \varepsilon, \Delta) \in P$ that determines the control action

$$u[t] = u_\varepsilon(t_i, Y[t_i], \varepsilon, \Delta), \quad t_i < t \leqslant t_{i+1} \tag{6.7}$$

is obtained from the condition of extremal displacement

$$(y[t_i] - z^{(c)}[t_i])' \begin{Vmatrix} B^*(t_i) u_e(t_i, Y[t_i], \varepsilon, \Delta) \\ \varphi(t_i, u_e(t_i, Y[t_i], \varepsilon, \Delta)) \end{Vmatrix} = \min_{v \in P} (y[t_i] - z^{(c)}[t_i])' \begin{Vmatrix} B^*(t_i) u \\ \varphi(t_i, u) \end{Vmatrix}$$

In accordance with (6.6) this means that vector $u_e(t_i, Y[t_i], \varepsilon, \Delta)$ is the solution of the problem for a minimum

$$l^{\circ\prime}[t_i] B^*(t_i) u_e(t_i, Y[t_i], \varepsilon, \Delta) + \varphi(t_i, u_e(t_i, Y[t_i], \varepsilon, \Delta)) = \min_{u \in P} [l^{\circ\prime}[t_i] B(t_i) u + \varphi(t_i, u)] \tag{6.8}$$

where $l^\circ[t_i] = l^\circ(t_i, Y[t_i], \varepsilon, \Delta)$ is the solution for a maximum of (6.5). This completely defines the step-by-step algorithm.

Hence, when it is necessary to organize a step-by-step control which is guaranteed by the inequality (6.1), one has to select a reasonably small parameter $\varepsilon > 0$ and a partitioning $\Delta\{t_i\}$ with a fairly small step $\delta > 0$. Let the partitioning $\Delta\{t_i\}$ contain k steps $(t_i, t_{i+1}]$, $i = 1, \ldots, k$. To calculate the control action $u[t]$ (6.7) it is necessary in the course of it to turn k times to the solution of the ancilliary problems (6.5) and (6.8). Each of these is a problem of convex programming.

A similar control procedure can be justified also in the case when the ancilliary problem of the form (6.5) is solved not for the function $\rho(t_i, Z[t_i], \Delta\{\tau_i\})$ but already for the limit function $\rho(t_i, Z[t_i])$. In that case the control action $u[t] = u^c(t_i, Y[t_i], \varepsilon) = u_e(t_i, Y[t_i], \varepsilon)$, $t_i < t \leqslant t_{i+1}$, determined by the condition of extremal displacement, does not depend on partitioning $\Delta\{t_i\}$, and is determined only by the instant t_i of realization of the transform $Y[t_i]$, and by the parameter ε. Then the vector $u^c(t_i, Y[t_i], \varepsilon)$ will be the value of the universal optimal strategy $u^c(t, Y, \varepsilon)$.

Note that in certain cases it is convenient when determining $\rho(\tau_*, Z[\tau_*], \Delta\{\tau_j\})$ in (2.8) and (2.9) to select not the norm $\| l(\cdot) \|$ of the form (2.7), but some other norm. The following norm often proves convenient:

$$\| l(\cdot) \| = (M\{| l(\omega) |^2\})^{1/2}$$

For instance, such a norm may be selected in the case, when the indicator γ (1.7) has the form

$$\gamma = | x[\vartheta] | \tag{6.9}$$

In that case we have in (1.7) $\varphi \equiv 0$ and $\psi \equiv 0$, and the previous construction can be simplified without introducing supplementary coordinates η and ζ. However there may appear in the reasoning details related to the fact that the solutions of the ancilliary problems of the form (6.5) may be non-unique. Hence it may be advisable also in the case of indicator γ (6.9) for determining the vector $l^\circ[t_i]$ to retain the complete scheme of calculations described above. In part of that scheme where the functions φ and ψ appear, the respective terms will, of course automatically disappear.

Instead of sequencies of random quantities ξ_j it is possible to select some probability process $\xi[\tau, \omega]$ continuous in time τ with idependent increments. For instance, it is possible to select a standard Brownian process. It is then possible to avoid the partitioning $\Delta\{\tau_j\}$, and the theoretical reasoning takes a more concise form. However in the case of a continuous process $\xi[\tau, \omega]$ the problem of the existence of the random maximizing element $l(\omega)$ proves to be more complicated, since the respective required martingale $l_*[\tau] = M\{l(\omega) \mid \{\xi[\nu, \omega], \tau_* \leqslant \nu < \tau\}$ appears to be less adapted to this ancilliary problem. We stress that the difficulty arises in connection with the existence of a completely random vector. The maximizing value $l^\circ[t] = l^\circ(t, Y[t], \varepsilon)$ for the vector l_* that plays the part of the expectation $M\{l(\omega)\}$ is also present in the selection of the continuous process $\xi[\tau, \omega]$.

The signal $q^*[t]$ can be replaced by some other information carrier. It is, thus, possible to use the information sets $G[t]$ obtained in some way; they are composed of phase states compatible with the current information.

For instance, in the case of the indicator γ (6.9) the theory set forth above can be reformulated without altering its essence in a clear manner in terms of the sets $G[t]$ composed of possible states $x[t]$. The part of the program-vectors $r(t, \omega)$ is transfered to the program-sets

$$N(\tau, \omega) = \left\{w : w = w_G - \int_{t_*}^{\tau} X[\tau, \nu] B(\nu) u(\nu, \omega) d\nu, \ w_G \in G(\tau, \omega), \ \omega \in \Omega, \ \tau_* < \tau \leqslant \vartheta \right\}$$

The respective formulae are automatically rewritten by introducing in appropriate places support functions $n(l, \tau, \omega)$ which are non-anticipatory on $\xi_j = \xi[\tau_j]$ and of sets $\mathcal{N}(\tau, \omega)$ (or sets with $\mathcal{N}(\tau, \omega)$). In that form the respective formulae are sometimes simplified even further.

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